Truesdell Lecture

I am deeply honored to join the list of those who have given the Truesdell Memorial Lecture, among whom was James Serrin, who recently died. Both Clifford Truesdell and Jim Serrin strongly influenced my career. I would like to give a couple of early nonscientific reminiscences of them.

In 1963 I took a course on the Calculus of Variations from Serrin. The serious part of this course formed the basis of my early research. But before getting serious, Serrin discussed the classical catenary problem. I went to Serrin's office to ask him how one could generalize the problem to a nonuniform nonlinearly elastic string. He thought quietly for a few minutes, and then announced to me that he could solve the generalized problem, but he never divulged to me how he would do it.

About that time Truesdell came to Minnesota. Many of us were eager to see this fierce expositor who was clarifying and generalizing continuum mechanics with his recently published Handbuch article with Toupin and with his sponsorship of Serrin's Handbuch masterpiece on fluids. Truesdell's first talk, subsequently published, debunked Leonardo da Vinci's status as a founder of the science of mechanics (a view Truesdell later modified after the appearance of the Madrid codices).

Truesdell's second talk was on his research on elastodynamics. For it, I was sitting behind my mentors Jim Serrin and Hans Weinberger. Early in the talk, Truesdell presented an open problem, saying that Serrin would no doubt solve it by the end of the lecture, just as he often had done as a graduate student. And sure enough, at the end of the lecture, Weinberger solved it. Truesdell later published the paper in Latin in the Archive, thanking Weinberger, who was latinized as Johannes Vineomontanus.

A year later, when I finished my thesis, Serrin said he would sign to approve it provided that he did not have to attend its defense. (Serrin was not lazy, but extremely busy. In his courses, he carefully graded all homework assignments with copious illuminating comments.) Serrin then arranged for Truesdell to get me a travelling fellowship to attend a CIME meeting in Bressanone organized by

Grioli and Truesdell. I first met Truesdell face-to-face at a cocktail party given by him and his wife Charlotte. It was memorable because they offered only two drinks: Punt and Mes.

I caught up with Truesdell while he was walking to a session and summoned up enough courage to describe my thesis to him. He strongly urged me to submit a polished version to him, which he communicated to the Archive.

Now, at the meeting I was astonished to see several Italian professors marching around leading triangular wedges of their young assistants, some of whom were armed with cigarette lighters to immediately assuage their professors' addictions. And these young assistants were equally astonished to see me, a newly hatched PhD, walking side by side with Truesdell in friendly conversation. In fact, Truesdell, who could be a pugnacious opponent of established scientists whose work Truesdell found unworthy, was noted for his encouragement, congeniality, and support to young scientists, of whom Serrin was one of the first to benefit.

Stuart S. Antman
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STABILITY THRESHOLDS IN MATERIAL RESPONSE

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In a series of papers beginning in 1947 Rivlin obtained explicit solutions of concrete problems for general classes of nonlinearly elastic solids and non-Newtonian fluids. His work inspired many others to study problems with such generality. Shortly thereafter, Truesdell promoted the use of general nonlinear constitutive equations with the philosophy of “generality in order to reach simplicity and clarity”.

This lecture is concerned with practical aspects of the use of general constitutive equations to detect thresholds in material behavior separating qualitatively different dynamical responses. This approach is based on agnosticism: We do not know the constitutive equations for many materials, biological and industrial, smart and dumb, old and new. (Indeed, the elastic moduli for the I-beams used in building construction have a 30% error due to residual stresses introduced during manufacture. This lecture, however, is concerned with nonlinear material response.)

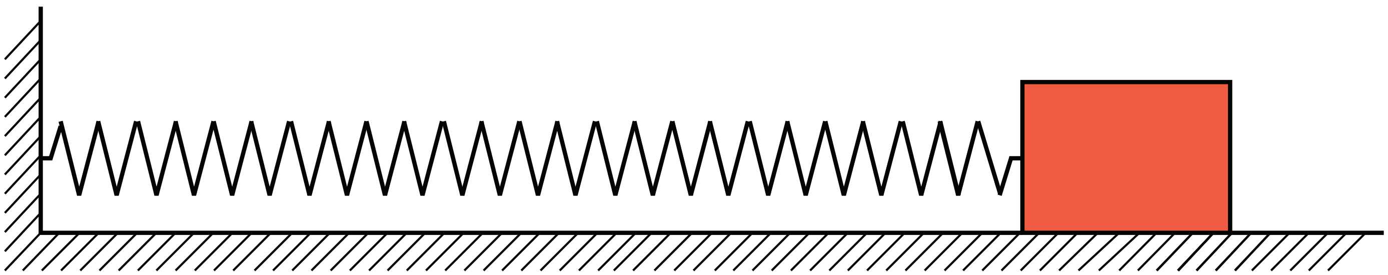
My philosophy:

In classical continuum mechanics always use exact Euclidean geometry (because it is very hard to prove that $\sin\theta = \theta$).

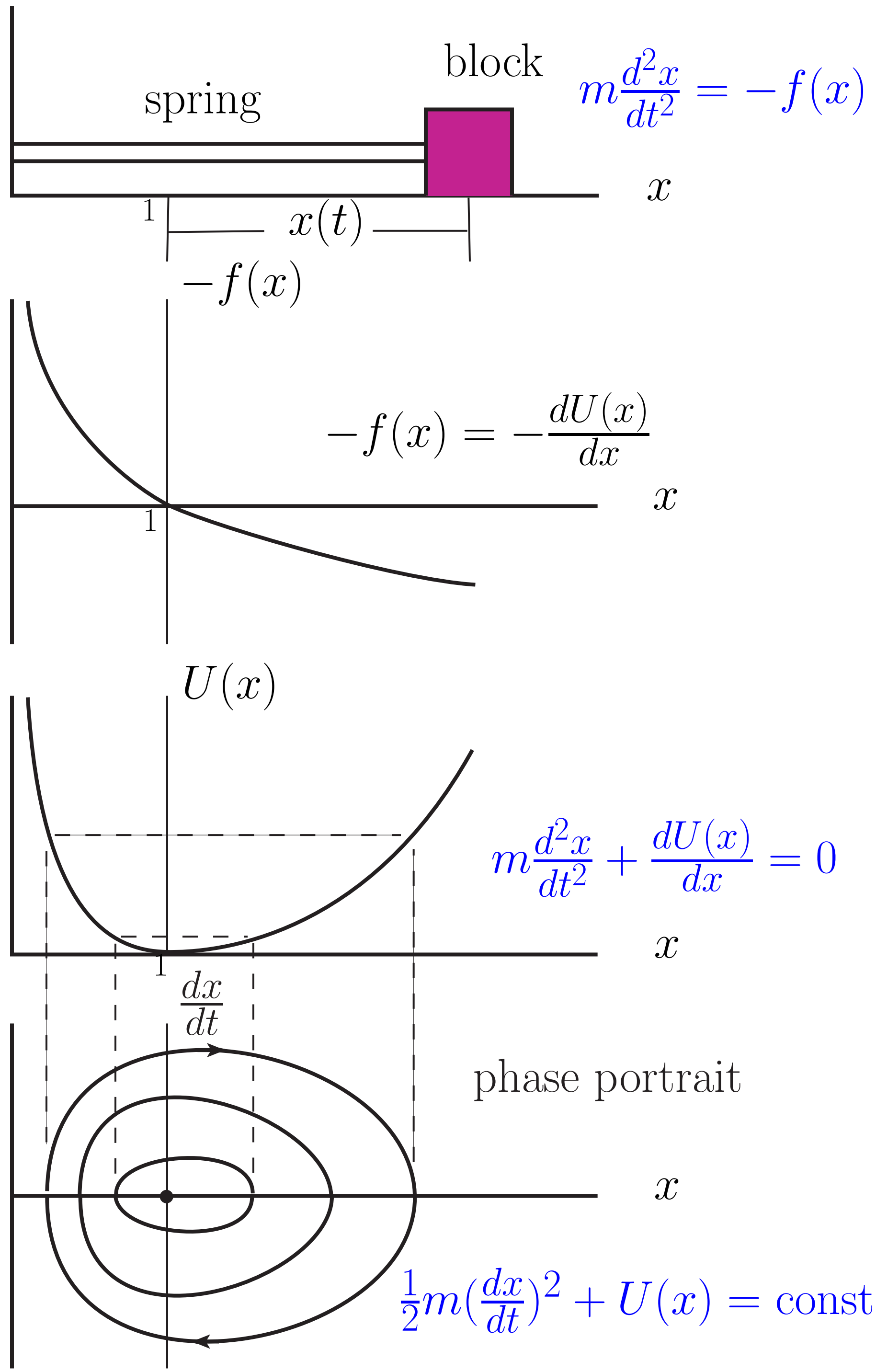
Always use the exact physical principles, like Newton's laws.

Then the only freedom available is in the choice of constitutive equations.

Start with the simplest problem of mechanics:

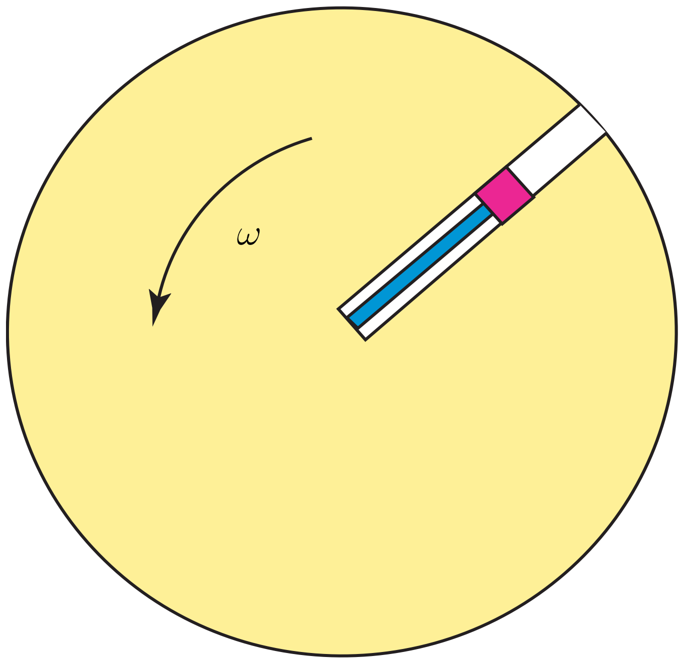


Mass point on a spring (with no friction)


block
spring
$m\frac{d^2x}{dt^2} = -f(x)$
x
1
$x(t)$
$-f(x)$
$-f(x) = -\frac{dU(x)}{dx}$
x
1
$U(x)$
$m\frac{d^2x}{dt^2} + \frac{dU(x)}{dx} = 0$
x
1
$\frac{dx}{dt}$
phase portrait
x
$\frac{1}{2}m(\frac{dx}{dt})^2 + U(x) = \text{const}$

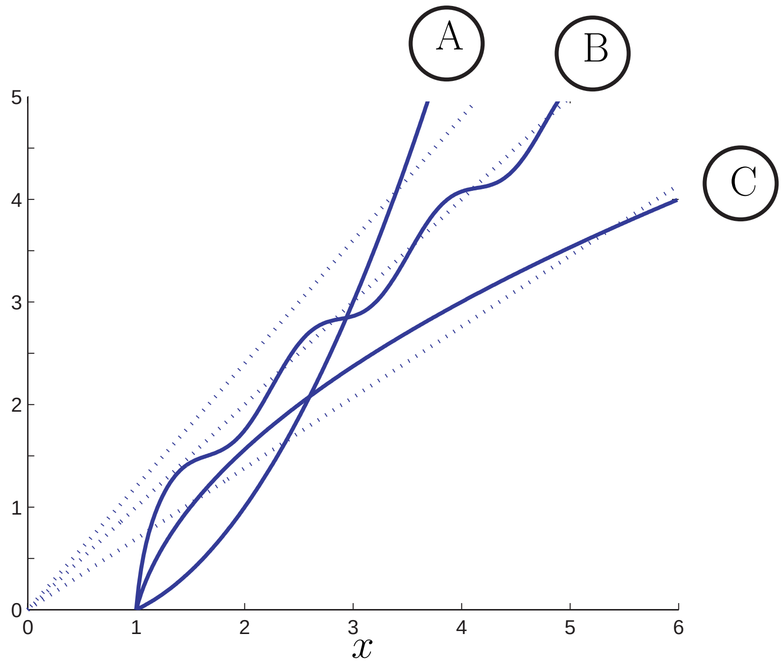
For this problem, the behavior of every spring is qualitatively independent of any constitutive function f with the indicated properties, so there are no thresholds. Make the problem a little more complicated:

A mass-spring system on a turntable



$$m\frac{d^2x}{dt^2} - m\omega^2 x + f(x) = 0,$$

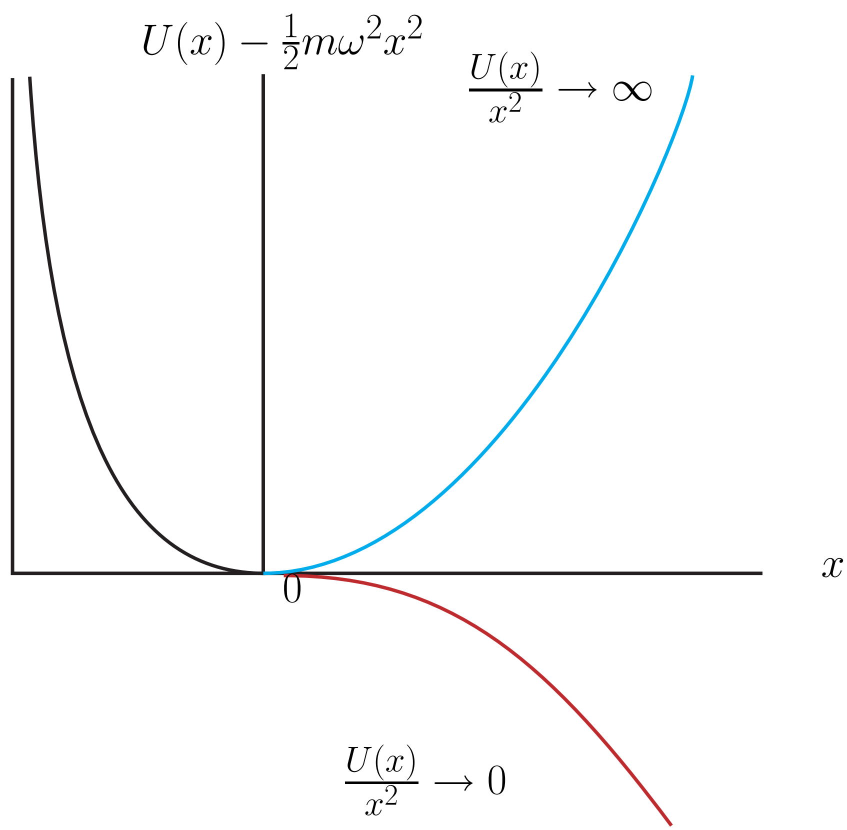
First look at steady solutions: $m\omega^2 x = f(x)$.



Graphs A, B, C of three constitutive functions f and graphs, with dotted lines, of the straight lines $x \mapsto m\omega^2 x$ for three different $m\omega^2$. Each intersection of a graph of f with a straight line corresponds to a steady-state of the spring. Graph A describes an asymptotically strictly superlinear f. For such a function there is at least one steady state for each ω^2. Graph C describes an asymptotically strictly sublinear f. For such a function there are at least two steady states for ω^2 less than a threshold value and no solutions for ω^2 exceeding that value. Graph B describes a function, which could be asymptotically linear, e.g., just linear, that has neither of the properties of A and C.

There is a rich range of possibilities for the dynamics, exhibited by the phase portraits, largely dictated by the behavior of the stored energy U in the energy equation

$$\frac{1}{2}m\left[\frac{dx}{dt}(t)\right]^2 + U(x(t)) - \frac{1}{2}m\omega^2 x(t)^2 = \text{const.}$$



For asymptotically strictly subquadratic U, there are solutions that are unbounded as $t \to \infty$ and there need not be any steady states. For asymptotically strictly superquadratic U, all solutions are bounded and there is always at least one steady state.

Constitutive functions f that are neither asymptotically strictly sublinear nor asymptotically strictly superlinear constitute a threshold. Included in this class of constitutive functions are the linear functions, beloved in traditional applied mechanics. The analysis of problems for constitutive functions in this threshold requires a "microscopic" analysis (examining elastic moduli): A problem with a specific constitutive function might misleadingly suggest that solutions for all materials are unstable or it might misleadingly and dangerously suggest that solutions for all materials are stable.

Analogous issues arise for solids subject to hydrostatic pressure, the potential energy of which is the pressure times the change in enclosed area (quadratic in the position field) for 2-dimensional problems or the the pressure times the change in enclosed volume (cubic in the position field) for 3-dimensional problems. In general, such thresholds in static problems can be expected for live loads, which have potentials that are not affine in the position.

Springs as Continua

There is a very rich theory for the motion of a mass point on a light spring, regarded as a 1-dimensional nonlinearly viscoelastic body with its mass proportional to a small parameter ε:

$$\varepsilon\rho(s)w_{tt} = \partial_s n(s, w_s, w_{st}) + f(s,t)$$
$$w(0,t) = 0, \quad m w_{tt} = -n(1, w_s(1,t), w_{st}(1,t)).$$

The reduced problem ($\varepsilon = 0$) is not governed by the traditional ordinary differential equation, but rather by an equation with memory. A rigorous and delicate asymptotic analysis shows (based on techniques of S. N. Berns(h)tein using the Maximum Principle) shows that the solution of the reduced This theory has been extended to the motion in space of a rigid body attached to a light nonlinear viscoelastic rod, governed by an 18th-order parabolic-hyperbolic system. (The justification using the Faedo-Bubnov-Galerkin method is easier.)

Threshold: $\varepsilon = 0$.

References

SA, The paradoxical asymptotic status of massless springs, *SIAM J. Appl. Math.* **48** (1988) 1319–1334.

SA, R. S. Marlow, & C. P. Vlahacos, The complicated dynamics of heavy rigid bodies attached to light deformable rods, *Quart. Appl. Math.* **56** (1998) 431–460.

SA & T. I. Seidman, Parabolic-hyperbolic systems governing the spatial motion of nonlinearly viscoelastic rods, *Arch. Ratl. Mech. Anal.* **175** (2005) 85–150.

SA & S. Ulusoy, The asymptotics of heavily burdened viscoelastic rods, *Quart. Appl. Math.*70 (2012) 437–467.

SA & J. P. Wilber, The asymptotic problem for the springlike motion of a heavy piston in a viscous gas, *Quart. Appl. Math.* **65** (2007) 471–498.

J. P. Wilber, Absorbing balls for equations modeling nonuniform deformable bodies with heavy rigid attachments, *J. Dyn. Diff. Eqs.* **19** (2002) 855–887.

J. P. Wilber, Invariant manifolds describing the dynamics of a hyperbolic-parabolic equation from nonlinear elasticity, *Dyn. Systems* **21** (2006) 465–490.

J. P. Wilber & SA, Global attractors for a degenerate partial differential equation from nonlinear viscoelasticity, *Physica D* **150** (2001) 179–208.

S.-C. Yip, SA, & M. Wiegner, The motion of a particle on a light viscoelastic bar: Asymptotic analysis of the quasilinear parabolic-hyperbolic equation, *J. Math. Pures Appl.* **81** (2002) 283–309.

Generalities on Elastostatics

$\boldsymbol{u}$ determines the position field.

$U(\boldsymbol{u})$ stored-energy functional for elastic body.

$\lambda V(\boldsymbol{u})$ potential energy functional for load. (E.g., $\lambda = \omega^2$ and λV is the potential of centrifugal force, or λ is the pressure and V is the change in area or change in volume.)

The Euler-Lagrange equations $\delta U + \lambda \delta V = 0$ for $U + \lambda V$ are the differential equations of elastostatics.

Fundamental Theorem of the Calculus of Variations: *A sequentially weakly lower semicontinuous functional on a bounded sequentially weakly closed nonempty subset of a reflexive Banach space attains its minimum there.*

A minimum of a functional Φ satisfying the "coercivity condition" $\Phi(u) \to \infty$ as $\|u\| \to \infty$, if it exists, lies in a bounded set.

Ignore the serious issue of regularity by presuming that a minimizer satisfies the Euler-Lagrange equation at least in some weak sense.

Suppose that $U(u) \to \infty$ as $\|u\| \to \infty$.

Is there a solution of the Euler-Lagrange equation $\delta U + \lambda \delta V = 0$? Nonexistence may signal dynamical behavior.

Is there a minimizer of $U + \lambda V$ for all λ? (Is there an equilibrium solution of a body under hydrostatic pressure for all pressures?) Yes, if U dominates V. (In this case a regular minimizer satisfies the Euler-Lagrange equation.) The U's that are comparable to V (i.e., that grow like V) form a threshold.

Is there a minimizer of U for all fixed values of (the amplitude) V? (Is there an equilibrium solution of a body under hydrostatic pressure for all enclosed volumes or areas?) Yes, for typical V's. (The Lagrange Multiplier Rule delivers the Euler-Lagrange equation.)

Is there a minimizer or maximizer of V for all fixed values of (the amplitude) U? (Is there an equilibrium solution of a body under hydrostatic pressure for all stored elastic energies?) Yes, for typical V's. (The Lagrange Multiplier Rule delivers the Euler-Lagrange equation.)

St. Venant-Kirchhoff Constitutive Equation

Let $\boldsymbol{F}$ be the (transposed) deformation gradient and let

$$\boldsymbol{E} := \tfrac{1}{2}(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{F} - \boldsymbol{I})$$

be the usual material strain tensor. Then the St. Venant-Kirchhoff constitutive equation (for a homogeneous isotropic body), which is used in many numerical computations, gives the second Piola-Kirchhoff stress $\boldsymbol{S}$ in the form

$$\boldsymbol{S} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$$

where λ and μ are the usual Lamé constants. The first Piola-Kirchhoff stress tensor is $\boldsymbol{T} = \boldsymbol{F}\boldsymbol{S}$. This constitutive equation has the virtue that it is an obvious generalization of that of linear elasticity.

Since $\boldsymbol{S}$ is quadratic in $\boldsymbol{F}$, it follows that $\boldsymbol{T}$ is cubic in $\boldsymbol{F}$, and the stored energy is quartic in $\boldsymbol{F}$. Thus this material is strong in resisting large $\boldsymbol{F}$ and is on the "safe side" of thresholds for problems with centrifugal forces or hydrostatic pressures. It can accordingly predict innocuous behavior whereas a real material could suffer dangerous instabilities. (Other defects: It does not penalize total compressions and does not satisfy the Strong Ellipticity Condition (so its wave propagation properties are not those of nice hyperbolic systems. Its energy, however, is not convex, so problems for it could admit bifurcation and buckling under dead loads).

Total Compression

The St. Venant-Kirchhoff Material does not prevent $\det \boldsymbol{F}$ from vanishing at material points or even becoming negative there (which could occur at corners). Numerical output in the form of colored graphics or movies may fail to show such a total compression or a change of orientation.

Mark Twain: There are lies, damned lies, and statistics.

Phil Colella: There are lies, damned lies, and colored graphics.

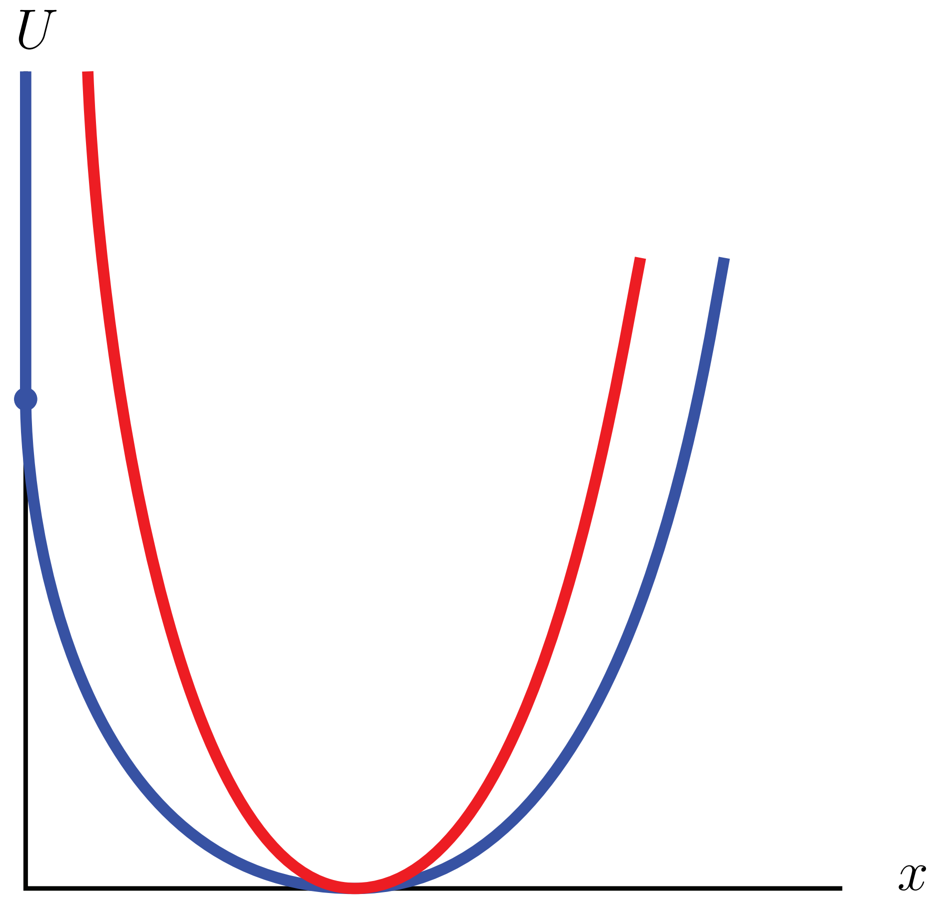
To try to construct a material that precludes a total compression, one could require that the stored-energy function $U \to \infty$ as $\det \boldsymbol{F} \searrow 0$ (and $U = \infty$ for $\det \boldsymbol{F} \leq 0$). Then an energy estimate could lead to

$$\int_{\mathcal{B}} U \, d\boldsymbol{x} < \infty,$$

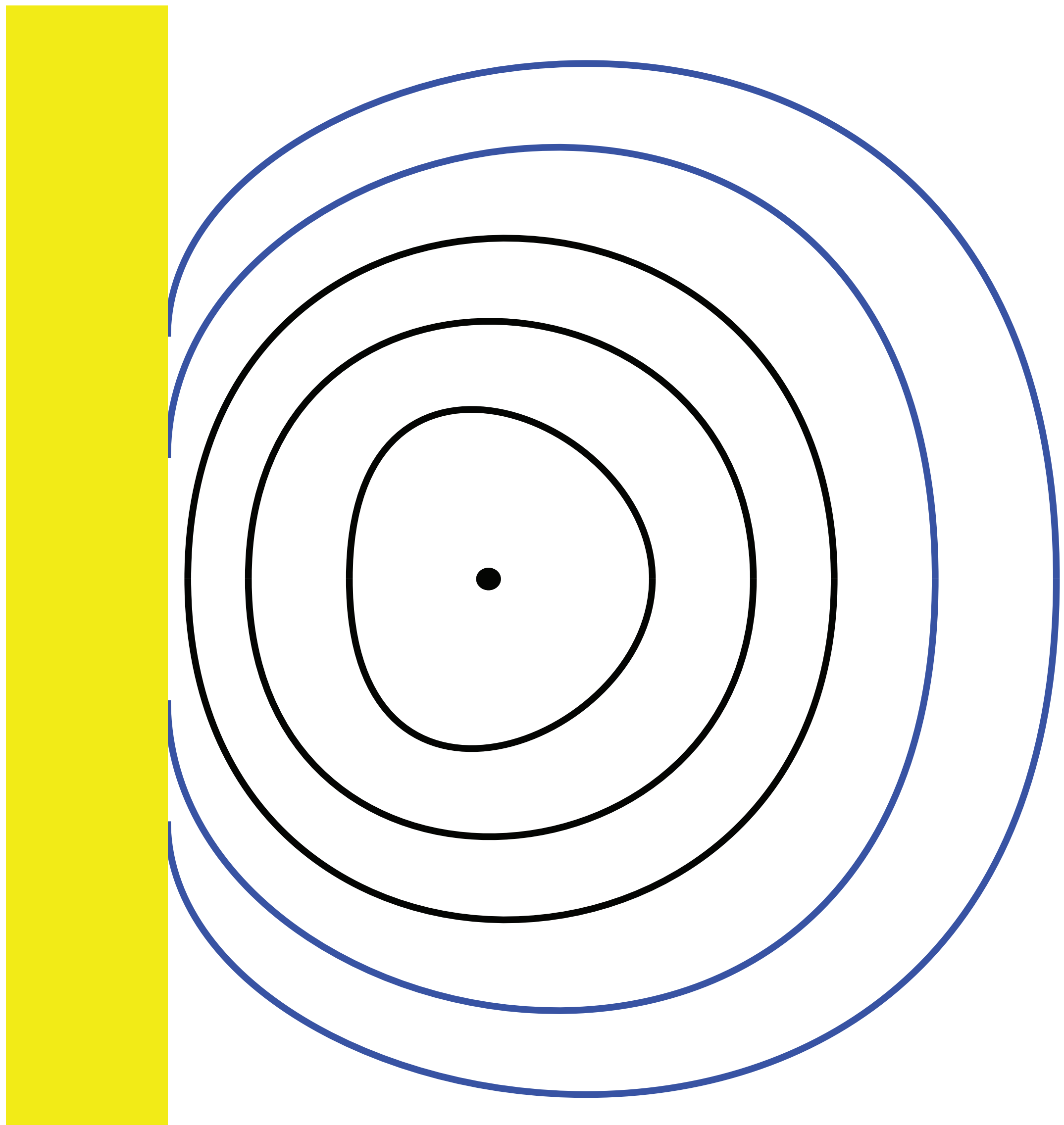
which would imply that $\det \boldsymbol{F}$ could vanish only on a set of spatial measure 0 for each time. For some equilibrium problems, a deeper analysis ensures that $\det \boldsymbol{F} > 0$ everywhere. But more is needed for dynamical problems. A hint for how to do this comes from the trivial problem of the longitudinal motion for the spring. The graphs already shown for it are (intentionally) ambiguous in that they do not indicate what happens at a total compression.

It is intuitively clear that an infinite force should be needed to effect a total compression. It is not so clear that an infinite stored energy should be needed.

Thus there are two classes of convex stored-energy functions; those that are unbounded and those that have bounded values at total compression. (There is no mathematical threshold separating these classes. What is important is that there are two classes.)



When the energy has a finite limit with slope $-\infty$ at total compression, the phase portrait does not determine the motion.



One way to determine a specific motion is to embed the constitutive equation in a family of constitutive equations with the negative spring force

$$f(x) + \varepsilon g(x, \dot{x})$$

where g accounts for an internal viscous friction and ε is a small positive parameter, the "viscosity". If the friction becomes infinite at total compression, i.e., if

$$g(x, \dot{x}) \to \infty \quad \text{as} \quad x \to 0,$$

then no phase-plane trajectory touches the line $x = 0$. The limit as $\varepsilon \searrow 0$ gives a unique and well-defined phase portrait for the problem (with a jump in the velocity: a zero-dimensional shock).

The importance of this trivial example is that the assumption that the viscosity becomes infinite at a total compression leads to the only available proofs that solutions of the partial differential equations of motion (of any order, e.g., 18) with but one space variable) for nonlinearly viscoelastic materials have solutions that never suffer a total compression anywhere. (It suffices for such problems that the stress become infinite at total compression, but the stored energy need not.)

These remarks suggest a provisional splitting (threshold) between materials with such viscosities and those without.

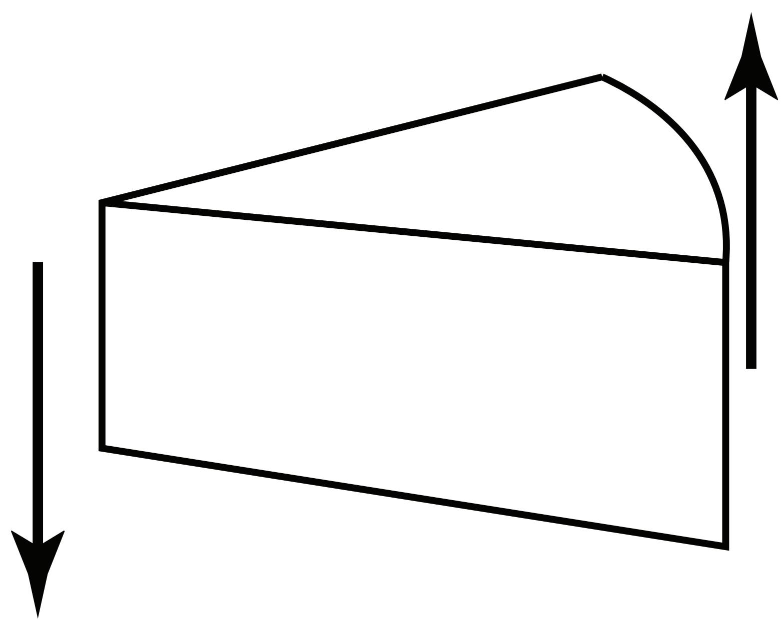
Singularities

Threshold effects can be due not only to live loads, but also to singular behavior in the body. A simple example: Semi-inverse deformation of a wedge $\mathcal{B}$ defined by cylindrical coordinates

$$0 \le r \le 1, \quad 0 \le \phi \le \alpha, \quad 0 \le z \le L$$

with the unit vector $\boldsymbol{k}$ along the "vertical" axis. The deformation is

$$(r, \phi, z) \to (r, \phi, z + h(r)), \qquad h(1) = 0.$$



Let the tractions on the faces $\phi = 0, \alpha$ be 0, let the traction on the cylindrical face be the constant $T\boldsymbol{k}$, but do not specify tractions on the top and bottom faces $z = 0, L$ so that they can be adjusted to ensure that the resultant force and torque on the body are zero. Let T_{31} be the first Piola-Kirchhoff stress on the cylindrical surface $r =$ const., acting in the $\boldsymbol{k}$-direction. One of the equilibrium equations is

$$\frac{d}{dr}(rT_{31}) = 0$$

so that

$$rT_{31} = T \quad \Longrightarrow \quad \lim_{r \to 0} rT_{31} = T.$$

Thus T_{31} is infinite on the edge $r = 0$ of the wedge.

Is $h(0)$ finite?

Let $\boldsymbol{F}$ be the (transposed) deformation gradient and let $\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$ be the right Cauchy-Green deformation tensor, so that $\operatorname{tr} \boldsymbol{C} = \|\boldsymbol{F}\|^2$. Suppose that the stored energy U satisfies

$$U \geq \text{const}(\operatorname{tr} \boldsymbol{C})^{p/2} - \text{const} \geq \text{const}|h'|^p - \text{const}, \quad p > 1.$$

Let the admissible solutions satisfy $\int_{\mathcal{B}} U\, dv \equiv \text{const} \int_0^1 rU\, dr < \infty$. Then

$$\begin{aligned}
|h(r)| &\leq \int_r^1 |h'(s)|\, ds = \int_r^1 s^{-\frac{1}{p}} s^{\frac{1}{p}} |h'(s)|\, ds \\
&\leq \left[\int_r^1 s^{-\frac{1}{p}\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}} \left[\int_r^1 |s^{\frac{1}{p}} h'(s)|^p\, ds\right]^{\frac{1}{p}} \quad \text{(Hölder)} \\
&\leq \text{const.} \left[\int_r^1 s^{-\frac{1}{p-1}} ds\right]^{\frac{p-1}{p}} = \text{const.} \left|1 - r^{\frac{p-2}{p-1}}\right|^{\frac{p-1}{p}}.
\end{aligned}$$

Thus $h(0)$ is bounded if $p > 2$.

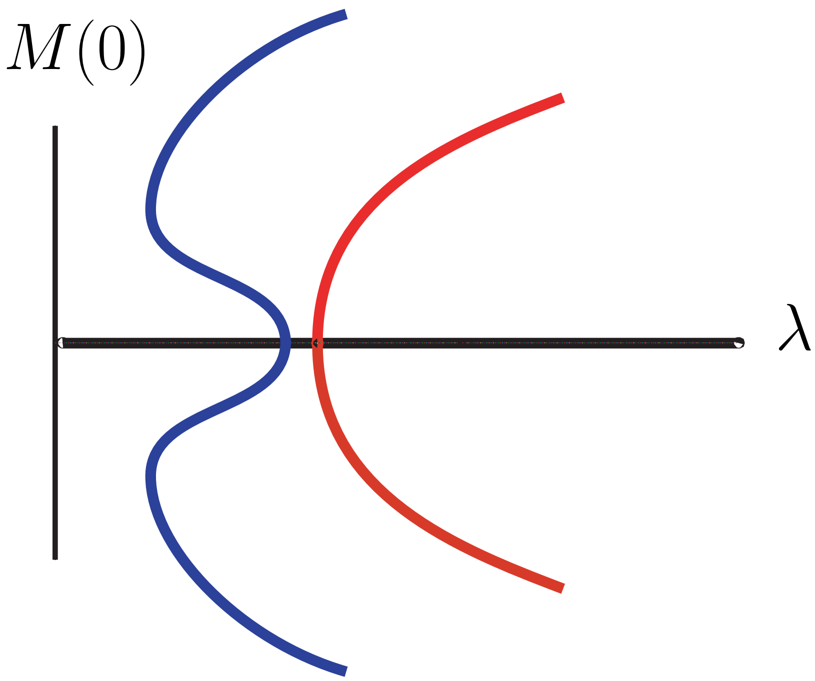
To see intuitively that $p = 2$ defines a threshold, suppose that $T_{31} \sim |h'|^{p-1}\text{sign}(h')$ and that h' is everywhere positive. Then $rT_{31} = T$ implies that

$$h' \sim r^{-1/(p-1)} \quad \Longrightarrow \quad -h(r) \sim \int_r^1 s^{-1/(p-1)}\, ds = \left.\frac{s^{\frac{p-2}{p-1}}}{\frac{p-2}{p-1}}\right|_r^1,$$

which $\to -\infty$ as $r \searrow 0$ if $p < 2$. (Linear response is threshold.)

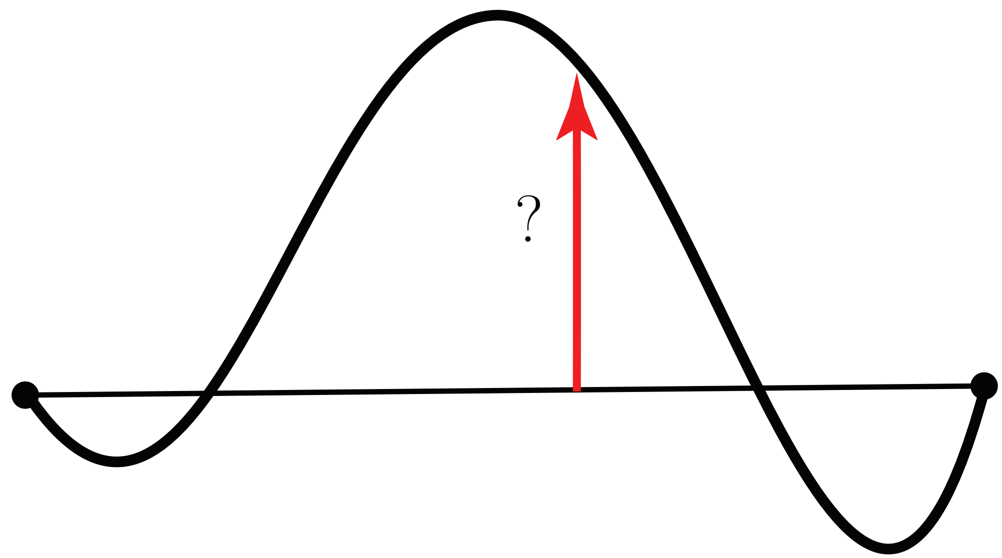
Inverse Problems

Buckling. The implications for stability of subcritical and supercritical bifurcation are very different and important: In the presence of imperfections the effective buckling load can be well below the smallest eigenvalue of the linearized problem. The threshold in constitutive functions distinguishing subcritical from supercritical bifurcation, which can readily be found, is not so far from the traditional linear constitutive equation. So its use can be dangerously misleading. (For global buckling problems, like Euler buckling of a nonlinearly elastic column, one actually can determine constitutive functions producing a given global bifurcating branch.)

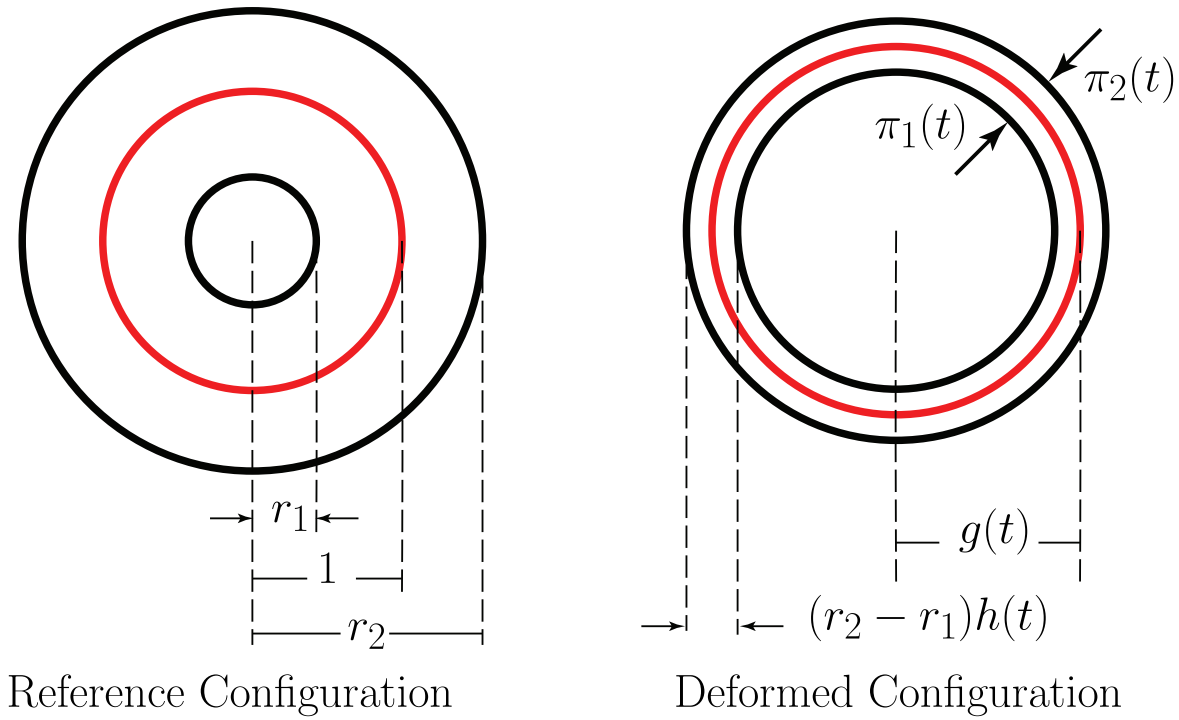

$M(0)$
λ

Modelling

Many "derivations" of the equations of motion for the vibration of an elastic string fixed tautly at its ends are based on the hypothesis that in such a motion each material point moves perpendicular to the line joining its ends. For what constitutive equations is this true? Answer: Only for very special linear constitutive equations, which occur with probability 0 (B. Fleishman & J. B. Keller).



Dynamics of Cylindrical and Spherical Shells



$$
\begin{aligned}
p_0(t) &:= \pi_1(t) - \pi_2(t), \\
p_1(t) &:= (r_1 - 1)\pi_1(t) - (r_2 - 1)\pi_2(t), \\
p_2(t) &:= (r_1 - 1)^2\pi_1(t) - (r_2 - 1)^2\pi_2(t), \\
p_3(t) &:= (r_1 - 1)^3\pi_1(t) - (r_2 - 1)^3\pi_2(t), \\
\mathbf{p}(t) &:= (p_0(t), p_1(t), p_2(t), p_3(t)),
\end{aligned}
$$

Pressure "pseudo-potentials":

$$\begin{aligned} \Pi^{(1)}(g,h,\mathbf{p}(t)) &:= \tfrac{1}{2}[p_0(t)g^2 + 2p_1(t)gh + p_2(t)h^2], \\ \Pi^{(2)}(g,h,\mathbf{p}(t)) &:= \tfrac{1}{3}[p_0(t)g^3 + 3p_1(t)g^2h + 3p_2(t)gh^2 + p_3(t)h^3], \end{aligned}$$

(1) for cylindrical shells, (2) for spherical shells. Equations of motion:

$$\begin{aligned} \rho A^{(n)}g_{tt} + \rho I^{(n)}h_{tt} + G^{(n)}(g,h,g_t,h_t) &= \partial_g \Pi^{(n)}(g,h,\mathbf{p}(t)), \\ \rho I^{(n)}g_{tt} + \rho J^{(n)}h_{tt} + H^{(n)}(g,h,g_t,h_t) &= \partial_h \Pi^{(n)}(g,h,\mathbf{p}(t)), \end{aligned}$$

$n = 1, 2$, a fourth-order semilinear system of ordinary differential equations with time-dependent coefficients. G and H are the constitutive equations for the generalized resultants corresponding to g and h.

Simplest dynamical problem: Elastic spherical shell under constant pressures: There is a stored-energy function $W(g,h)$ such that

$$G = W_g, \qquad H = W_h.$$

Define an amplitude function and the kinetic energy function:

$$\begin{aligned} \Phi(g,h) &:= \tfrac{1}{2}(\rho A g^2 + 2\rho I gh + \rho J h^2), \\ K(\dot g,\dot h) &:= \tfrac{1}{2}(\rho A \dot g^2 + 2\rho I \dot g\dot h + \rho J \dot h^2). \end{aligned}$$

Energy equation:

$$K(g_t, h_t) + W(g, h) - \mathit{\Pi}(g, h, \mathbf{p}) + \int_0^t \mathit{\Pi}(g, h, \mathbf{p}_t)\, d\tau$$
$$= E := K(g_t(0), h_t(0)) + W(g(0), h(0)) - \mathit{\Pi}(g(0), h(0), \mathbf{p}(0)).$$

This energy bound and the Gronwall inequality yield

Theorem. *Let the material of a spherical shell be strong in resisting extension in the sense that W is supercubic in g and h, so that there is a positive constant C such that*

$$C[W^{(2)} + 1] \geq \mathit{\Phi}^{3/2}.$$

Then the solution of any initial-value problem exists for all time.

The energy equation implies that

$$\mathit{\Phi}_{tt} = 2K - G^{(2)}g - H^{(2)}h + 3\mathit{\Pi}^{(2)}$$
$$= 5K + 3W^{(2)} - G^{(2)}g - H^{(2)}h - 3E + 3\int_0^t \mathit{\Pi}^{(2)}(g, h, \mathbf{p}_t)\, d\tau.$$

Assume that the static pressures have so large an inflational effect that $E < 0$ and that the equilibrium response in tension is weak in resisting hydrostatic inflation in the sense that it is subcubic:

$$3W \geq Gg + Hh. \tag{1}$$

Then

$$\Phi\Phi_{tt} \geq \tfrac{5}{4}{\Phi_t}^2 \quad \Longrightarrow \quad \Phi(t) \geq \frac{\Phi(0)^5}{[\Phi(0) - \frac{1}{4}\Phi_t(0)t]^4}.$$

Theorem. *Under these conditions there are initial conditions for which the solution blows up in finite time for initial conditions with $\Phi_t(0) > 0$.*

Cylindrical shells behave differently. Weak shells can blow up, but only at infinity. There are a host of blowup thresholds for time-periodic pressures and for both spherical and cylindrical viscoelastic shells and there are various conditions for which viscoelastic shells have solutions having the same period as that of "compressive" pressures.

Parametric Resonance

Periodically forced cylindrical and spherical motions of cylindrical and spherical shells, not accounting for thickness strains. (A complicated trivial problem; the nontrivial problem is that of the loss of cylindricity or sphericity.)

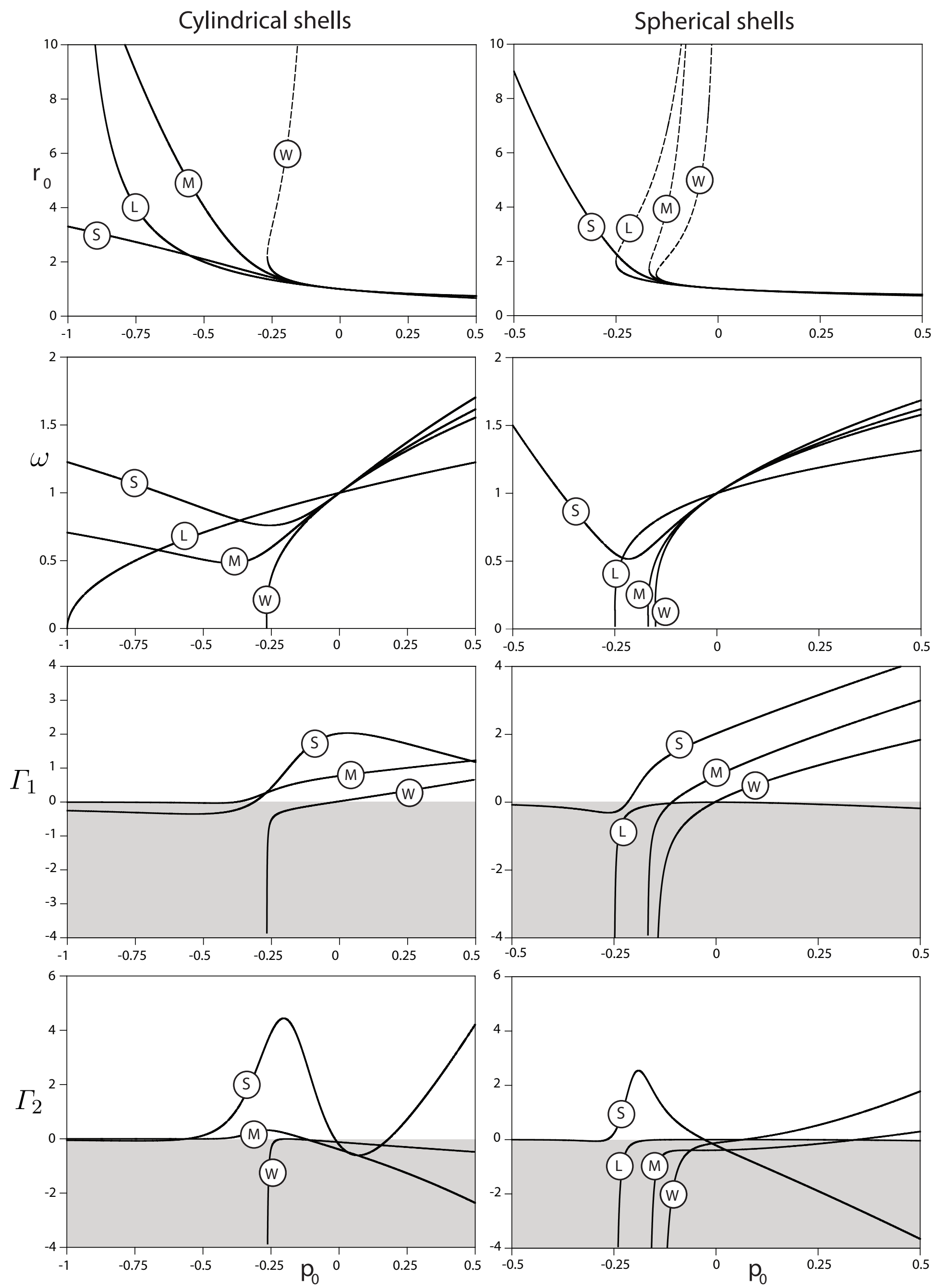
$$r_{tt} + G(r) + \tfrac{1}{2}\varepsilon^2 S(r, r_t) + [p_0 + \varepsilon^\beta P \cos \Omega t] r^\nu = 0, \qquad \nu = 1, 2.$$

p_0 is positive in compression. Let ω be the natural frequency of the equation linearized around the equilibrium state under pressure p_0. For primary resonance take $\beta = 3$. Then periodic motions have the backbone curve

$$\Omega(\varepsilon) \sim \omega + \frac{\varepsilon^2}{8} \Gamma_1 a^2 \quad \text{where } a \text{ is an amplitude.}$$

Four constitutive equations:
W (weak): $G \sim r^{1/2}$ for r large,
L (linear): G is linear in r,
M (moderate): $G \sim r^{3/2}$ for r large,
S (strong): $G \sim r^{5/2}$ for r large.

M is strong for cylindrical shells and weak for spherical shells.



Cylindrical shells
Spherical shells
r_0
ω
Γ_1
Γ_2
p_0
p_0
S
L
M
W

Other Problems

A circular disk could have radially disposed strengthening fibers (as in G. I. Taylor's model for a parachute) or could have radially disposed fissures, producing aeolotropy with a singularity at the center. In the former case, the slightest pressure on the body produces infinite stress at the center. In the latter case, every pressure on the body produces zero stress at the center. Very slight changes from constitutive equations for isotropic media can produce a wide range of surprising behavior.

A pendulum restrained by a torsional spring with viscous friction is subject to a force f with $f(t) \nearrow \infty$ as $t \to \infty$. The pendulum starts our with any initial conditions. Every motion converges asymptotic to a vertical state. But if the damping is not large enough, the velocity is unbounded as for the function

$$t^{-1/4} \sin t^{3/2}.$$

Dynamics of Nonlinearly Elastic and Plastic Bodies

The equations of motion of nonlinear elasticity and plasticity form quasilinear hyperbolic systems, which admit shocks (discontinuities in velocity or strain). (Degenerate case: compressible inviscid gas.)

Admissibility or entropy conditions are adjoined to the governing (weak form of the) partial differential equations to ensure that (weak) solutions can be uniquely continued beyond the appearance of shocks. Various admissibility conditions have been proposed on the basis of mathematical convenience, physical (thermodynamical) insight, numerical intuition, and formal limiting processes as dissipative and regularizing terms in the constitutive equations go to zero. The resulting admissibility conditions are not equivalent.

The theory of such systems is rich when there is only one independent spatial variable, very rich when the order of these systems is 2, and extremely rich when the order of these systems is 3.

I favor admissibility conditions that are the traces of disappearing dissipative and regularizing terms (because this approach seems most physical. But there are serious mathematical difficulties.)

There are also continuum mechanical difficulties: The are simply formulated exact problems for nonlinearly viscoelastic layers (with general constitutive laws) that fail to produce any convenient admissibility conditions by this method.

Are there natural constitutive responses leading to natural shock structure?

An obstacle: The main dissipative mechanism in gas dynamics is the viscosity, which is possibly supplemented by thermal diffusivity. The spatial (= Eulerian) formulation of the Navier-Stokes equations governing the 1-dimensional longitudinal motion of a compressible, barotropic, viscous gas (in a cylinder) are

$$\varrho_t = -(\varrho v)_y,$$
$$\varrho(v_t + v v_y) = -p(\varrho)_y + \varepsilon v_{yy}$$

where the small positive parameter ε is the viscosity and $v_{yy} = \Delta v$ is the Laplacian of the velocity. Many numerical schemes for shocks, starting with that of Lax and Friedrichs, are based on difference schemes for a modified system like

$$\varrho_t = -(\varrho v)_y + \varepsilon \varrho_{yy},$$
$$\varrho(v_t + v v_y) = -p(\varrho)_y + \varepsilon v_{yy}.$$

These schemesare designed for any set of equations. But solid mechanics is usually formulated in material (Lagrangian) coordinates, and the tranformed version of the spatial Laplacian is not a Laplacian. For multi-dimensional problems the use of a Laplacian in a material formulation causes the the constitutive equations in the governing equations to cease to be invariant under observer, and therefore physically unrealistic and likely to lead to serious numerical error for rapidly rotating bodies. Invariant versions of these constitutive equations are available, and useful for removing errors. Consequences for shock structure?

References

SA, Existence and nonuniqueness of axisymmetric equilibrium states of nonlinearly elastic shells, *Arch. Ratl. Mech. Anal* **40** (1971) 329–371.

SA, The theory of rods, *Handbuch der Physik VIa/2*, Springer-Verlag, 1972, pp. 641–703.

SA, Regular and singular problems for large elastic deformations of tubes, wedges, and cylinders, *Arch. Ratl. Mech. Anal* **83** (1983) 1–52. Corrigenda, ibid. **95** (1986) 391–393.

SA & C. L. Adler, Design of material properties that yield a prescribed global buckling response, *J. Appl. Mech.* **54** (1987) 263–268.

SA & P. V. Negrón-Marrero, The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies, *J. Elasticity* **18** (1987) 131–164.

SA & R. Malek-Madani, Travelling waves in nonlinearly viscoelastic media and shock structure in elastic media, *Quart. Appl. Math.* **46** (1988) 77–93.

SA, A zero-dimensional shock, *Quart. Appl. Math.* **46** (1988) 569–581.

SA & J. F. Pierce, The intricate global structure of buckled states of compressible columns, *SIAM J. Appl. Math.*, **50** (1990) 395–419.

SA, Invariant dissipative mechanisms for the spatial motion of rods suggested by artificial viscosity, *J. Elasticity* **70** (2003) 55–64.

SA & W. Lacarbonara, Forced radial motions of nonlinearly viscoelastic shells, *J. Elasticity* **96** (2009) 155–190.

W. Lacarbonara & SA, Parametric instabilities of the radial motions of nonlinearly viscoelastic shells under pulsating pressures, *Int. J. Nonlin. Mech.*, **47** (2012) 461–472.

SA, Real artificial viscosity, preprint.