Introduction: Truesdell Lecture, 2008

I feel honored to be in the position as the Truesdell Lecturer and I appreciate the invitation and the expression of confidence from the Society. It is satisfying, indeed, to be in the company of Noll, Serrin and Coleman as the first three such inaugural Lecturers.

Please permit me, first, a few personal words: I finished my undergraduate studies in Mechanical Engineering in 1959, two years after Sputnik was launched. The USA had begun to believe that it was falling behind Russia in Science, Engineering and Mathematics. So, in 1959 the United States passed its first National Defense Act. One of its actions was to provide full support to a few US students for Graduate School Studies in Engineering, Science and Mathematics. It was a perfect time to move on to graduate studies as it was the beginning of the great period of scientific discovery and development that took place in the second half of the 20th century. There was support for fundamental interests and studies and it was the beginning of a "golden period" in research.

By that time, Clifford Truesdell, a man of mathematics, science and natural philosophy, of immense creativity who taught us to preserve scholarship, to question foundations and to follow a path of reason with principle, purpose and passion, had recently published his landmark article on the "Mechanical Foundations of Elasticity and Fluid Dynamics", which brought life and beauty to the field of continuum mechanics and stimulated interest in foundational matters throughout the community for years to come.

I was awarded a National Defense Act Scholarship for the study of Applied Mathematics at Brown University.

The Division of Applied Mathematics had a faculty that was second to none and the Division of Engineering was equally impressive. New research reports were produced and circulated daily. It was exciting and it felt like it was the center of all that was developing in mechanics. All kinds of visitors came from Europe and the US to study and to give courses and lectures.

At that time, it was considered bad taste by some at Brown to be studying Truesdell and the newly produced thesis of Walter Noll on the "Continuity of Solid and Fluid States", but some of us did so anyway. I was fortunate to share an office with fellow graduate student Mort Gurtin and we found much interest in the ideas and directions set out in these works.

In 1962, the thermodynamics of continua was being presented in a new and different way. It appeared to be more of an exact science built upon definite principles. And the second law was being made more tractable—it was being presented as something more fixed and unchanging–quite different than the usual verbal statements that were taught. Results in thermodynamics were beginning to be deduced from this new way of thinking. In a remarkable set of papers in the 1960's and 70's that streamed from the collaboration of Coleman and Noll, a theory for the thermomechanics of continua was established and began to be applied. It faced stiff resistance from the classical equilibrium-based thermodynamicists—especially the idea that the concept of entropy (and temperature, as well) may be applied outside of equilibrium, or away from near equilibrium situations. The importance of the Clausius-Duhem inequality in this work, a 2nd law restriction for rational continuum thermodynamics, certainly was under-appreciated by the "old guard", to say the least.

This defined the environment of my initial engagements with Clifford Truesdell and the related school of thought.

I have admired Clifford Truesdell's contributions over many years for their scholarship quality, their fundamental significance to the advancement of the physical and mathematical science of continuum mechanics and thermodynamics, their insight, their breadth, their objectiveness, their vitality and their novelty. His works have influenced my thinking in fundamental ways and for that I remain greatly indebted.

I have chosen to talk here about "Continuum Thermomechanics". Much of what I have to say is not new, modulo a few wrinkles. It may be considered a reflection on one aspect of the "golden period of research" of the last half century. Many have worked on various aspects of this topic over the years, including: Beatty, Gurtin, Noll, Podio-Guidugli, Šilhavý, ...

Continuum Thermodynamics from the Perspective of Invariance

Roger Fosdick

September, 2008

Body and its Configurations

Definition 1 *A* material body $\mathcal{B}$ *is a compact three dimensional Riemannian manifold.*

(i) $\mathcal{B}$ is an open set that can be locally parameterized by $\mathbb{R}^3$. Locally, the points $p \in \mathcal{B}$ associate with a rank-3 mapping $p = \bar{p}(X^1, X^2, X^3)$ which defines a coordinate basis through

$$\mathbf{b}_i(p)|_{p=\bar{p}(X^1,X^2,X^3)} := \frac{\partial}{\partial X^i}\bar{p}(X^1, X^2, X^3)\,.$$

(ii) A Riemannian metric for $\mathcal{B}$ is assigned such that for each local parametrization

$$g_{ij}(X^1, X^2, X^3) = \mathbf{b}_i(p) \cdot \mathbf{b}_j(p)|_{p=\bar{p}(X^1,X^2,X^3)}\,.$$

One may conceive of the possibility that the structure of the body can change with time, especially if the body is to be subject to loading and heating. The local metric and connection properties of the manifold may evolve coincidentally and in sympathy with the local physical environment of its particles during a thermodynamic process. This would mean the incorporation of more physics and constitutive structure into the classical continuum mechanics understanding of the body and its environment and could serve as an appropriate setting in which the novel theories of configurational forces and actions, developed in recent years, might be embedded. A theory for the generation of dislocations and other defect structures is in the minds eye.

Definition 2 *Given an embedding* $\boldsymbol{\kappa} : \mathcal{B} \mapsto \mathbb{E}^3$*, the set* $\mathcal{B}_{\boldsymbol{\kappa}} \equiv \boldsymbol{\kappa}(\mathcal{B}) \subset \mathbb{E}^3$ *is called the* reference configuration *of* $\mathcal{B}$*:*

$$\mathbf{X} \equiv \boldsymbol{\kappa}(p) \in \mathbb{E}^3 \quad \forall p \in \mathcal{B}$$

denotes the referential material point *in* $\mathbb{E}^3$ *of a material particle in the body* $\mathcal{B}$*.*

(i) A motion of $\mathcal{B}$ is a smooth time-parameter sequence of embedding's of $\mathcal{B}$ into $\mathbb{E}^3$ $\forall\ p \in \mathcal{B}, \ \ \forall\ t \in \mathbb{R}^+$:

$$\mathbf{x} = \boldsymbol{\chi}(p,t) = \boldsymbol{\chi}(\boldsymbol{\kappa}^{-1}(\mathbf{X}),t) := \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X},t)\,,$$

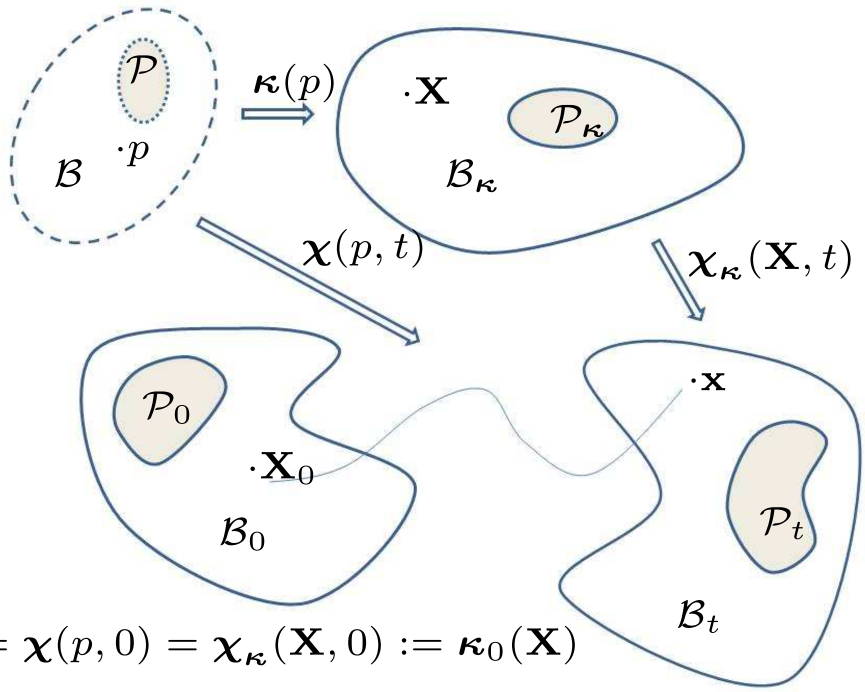
$\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t)$, the motion relative to $\mathcal{B}_{\boldsymbol{\kappa}}$, is assumed to be rank 3 so that $\det \nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t) \neq 0$ for all $t \in \mathbb{R}^+$.

(ii) The initial configuration $\mathcal{B}_0$ of $\mathcal{B}$ under the motion $\boldsymbol{\chi}$ is given by

$$\mathbf{X}_0 = \boldsymbol{\chi}(p,0) = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X},0) := \boldsymbol{\kappa}_0(\mathbf{X}) : \mathcal{B}_{\boldsymbol{\kappa}} \mapsto \mathcal{B}_0$$

and the motion relative to $\mathcal{B}_0$ is

$$\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\boldsymbol{\kappa}_0^{-1}(\mathbf{X}_0),t) := \boldsymbol{\chi}_0(\mathbf{X}_0,t)\,.$$



$$\mathbf{X}_0 = \boldsymbol{\chi}(p,0) = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X},0) := \boldsymbol{\kappa}_0(\mathbf{X})$$

Two common deformation gradients:

$$\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t) : \mathcal{B}_{\boldsymbol{\kappa}} \mapsto \text{Inv} \quad \text{and} \quad \nabla\boldsymbol{\chi}_0(\cdot,t) : \mathcal{B}_0 \mapsto \text{Lin}^+ .$$

The chain rule $\Longrightarrow$

$$\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t) = \nabla\boldsymbol{\chi}_0(\cdot,t)\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,0) = \nabla\boldsymbol{\chi}_0(\cdot,t)\nabla\boldsymbol{\kappa}_0(\cdot) .$$

Note that while $\det\nabla\boldsymbol{\chi}_0(\cdot,t) > 0$,

$$\text{sgn}(\det\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t)) = \text{sgn}(\det\nabla\boldsymbol{\kappa}_0(\cdot)) .$$

The elements of material volume in $\mathcal{B}_{\boldsymbol{\kappa}}$, $\mathcal{B}_0$ and $\mathcal{B}_t$ are related by

$$dv = |\det\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}|dv_{\boldsymbol{\kappa}} = (\det\nabla\boldsymbol{\chi}_0)dv_0$$

with

$$dv_0 = |\det\nabla\boldsymbol{\kappa}_0|dv_{\boldsymbol{\kappa}} .$$

The Fields

Mass

The mass density field $\rho : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^{+} \mapsto (0, \infty)$ denotes the Lagrangian form of the mass per unit volume of $\mathcal{B}_t$ at the referential material point $\mathbf{X}$ at time t. The total mass of $\mathcal{P}_t$ is

$$\mathcal{M}(\mathcal{P}, t) := \int_{\mathcal{P}_t} \rho \, dv \, .$$

Alternatively,

$$\mathcal{M}(\mathcal{P}, t) = \int_{\mathcal{P}_{\boldsymbol{\kappa}}} \rho |\det \nabla \boldsymbol{\chi}_{\boldsymbol{\kappa}}| \, dv_{\boldsymbol{\kappa}} = \int_{\mathcal{P}_0} \rho (\det \nabla \boldsymbol{\chi}_0) \, dv_0$$

where in the first integral ρ refers to $\rho(\mathbf{X}, t)$ while in the second ρ refers to $\rho(\boldsymbol{\kappa}_0^{-1}(\mathbf{X}_0), t)$.

Fundamental Kinetic Fields

In classical continuum mechanics, there are two kinetic fields which describe the transfer of force on the material points p of the body $\mathcal{B}$. These act on the configuration $\mathcal{B}_t$ during a motion:

1. The contact traction field $\mathbf{t} : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \times S^2 \mapsto \mathbb{E}^3$, represents the force per unit area $\mathbf{t_n}(\mathbf{X}, t) := \mathbf{t}(\mathbf{X}, t, \mathbf{n})$ acting at a point $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t)$ in $\mathcal{B}_t$ on an oriented material surface at $\mathbf{x}$ whose unit normal is $\mathbf{n}$. This is due to the interaction of adjacent material. Units are FL^{-2}.

2. The body force field $\mathbf{b} : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto \mathbb{E}^3$ represents the Lagrangian form of the force per unit mass, $\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$, acting at a point $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t)$ in $\mathcal{B}_t$ due to causes outside $\mathcal{B}_t$. Units are FM^{-1}.

Fundamental Thermodynamic Fields

Consideration of thermodynamics within the context of mechanics requires at least the presence of five additional basic thermodynamic fields:

1. The absolute temperature field $\theta : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto (0, \infty)$. Units are ^{o}A.

2. The specific internal energy field $\varepsilon : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto \mathbb{R}$. Units are FLM^{-1}.

3. The specific internal entropy field $\eta : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto \mathbb{R}$. Units are FLM^{-1}^{o}A^{-1}.

4. The contact heating field $q : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \times S^2 \mapsto \mathbb{R}$. This represents the rate of heat conduction per unit area, $q_{\mathbf{n}}(\mathbf{X}, t) := q(\mathbf{X}, t, \mathbf{n})$, acting at a point $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t)$ in $\mathcal{B}_t$ on an oriented material surface at $\mathbf{x}$ whose unit normal is $\mathbf{n}$. It is a surfacial contact interaction of adjacent material and the convention is: $q_{\mathbf{n}} > 0$ ($q_{\mathbf{n}} < 0$) corresponds to heat being transmitted across the surface in a direction *obtuse* (*acute*) to $\mathbf{n}$. Units are FLT^{-1}L^{-2}.

5. The specific radiant heating field $r : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto \mathbb{R}$. This represents the Lagrangian form of the rate of heating per unit mass $r = r(\mathbf{X}, t)$ acting at a point $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t)$ in $\mathcal{B}_t$. Units are FLT^{-1}M^{-1}.

Thermodynamic Processes

Definition 3 (Thermodynamic Process) *Given $\mathcal{B}$ and $\mathcal{B}_{\boldsymbol{\kappa}} := \boldsymbol{\kappa}(\mathcal{B})$. An ordered set of the nine fundamental fields*

$$\pi(\mathbf{X}, t) := \{\boldsymbol{\chi}_{\boldsymbol{\kappa}},\ \rho,\ \theta,\ \varepsilon,\ \eta,\ \mathbf{t}_{\mathbf{n}},\ \mathbf{b},\ q_{\mathbf{n}},\ r\}\,(\mathbf{X}, t)\,,$$

for all $\mathbf{X} \in \mathcal{B}_{\boldsymbol{\kappa}}$ and $t \in \mathbb{R}^+$, is called a thermodynamic process for *$\mathcal{B}$ if for any unit vector $\mathbf{n}$, any material part $\mathcal{P} \subset \mathcal{B}$, and for all $t \in \mathbb{R}^+$, the following two conditions are satisfied:*

1. *The balance of energy holds,*

$$\frac{d}{dt}E(\mathcal{P}, t) + \frac{d}{dt}K(\mathcal{P}, t) = P(\mathcal{P}, t) + Q(\mathcal{P}, t)\,;$$

2. *The Clausius-Duhem inequality holds,*

$$\frac{d}{dt}H(\mathcal{P}, t) \geq S(\mathcal{P}, t)\,.$$

Global Integral Measures for $\pi(\mathbf{X}, t)$

The total internal energy of $\mathcal{P}_t$ is

$$E(\mathcal{P}, t) := \int_{\mathcal{P}_t} \rho\, \varepsilon\, dv\,.$$

The total kinetic energy of $\mathcal{P}_t$ is

$$K(\mathcal{P}, t) := \int_{\mathcal{P}_t} \frac{1}{2} \rho\, |\mathbf{v}|^2\, dv\,.$$

The total mechanical working (i.e., power) of $\mathcal{P}_t$ is

$$P(\mathcal{P}, t) := \int_{\partial \mathcal{P}_t} \mathbf{t_n} \cdot \mathbf{v}\, da + \int_{\mathcal{P}_t} \rho \mathbf{b} \cdot \mathbf{v}\, dv\,.$$

The total heat working of $\mathcal{P}_t$ is

$$Q(\mathcal{P}, t) := \int_{\partial \mathcal{P}_t} q_{\mathbf{n}}\, da + \int_{\mathcal{P}_t} \rho\, r\, dv\,.$$

The total internal entropy of $\mathcal{P}_t$ is

$$H(\mathcal{P}, t) := \int_{\mathcal{P}_t} \rho \eta\, dv\,.$$

The total entropy flux for $\mathcal{P}_t$ is

$$S(\mathcal{P}, t) := \int_{\partial \mathcal{P}_t} \frac{q_{\mathbf{n}}}{\theta}\, da + \int_{\mathcal{P}_t} \rho \frac{r}{\theta}\, dv\,.$$

Translation Invariance and its Consequences

Axiom 1 (Translation Invariance) *Let* $\mathcal{B}$ *and* $\mathcal{B}_{\boldsymbol{\kappa}} := \boldsymbol{\kappa}(\mathcal{B}_{\boldsymbol{\kappa}})$ *be given with* $\mathbf{X} \in \mathcal{B}_{\boldsymbol{\kappa}}$*. If* $\pi(\mathbf{X}, t)$ *is a thermodynamic process for* $\mathcal{B}$*, then* $\pi^{\lambda}(\mathbf{X}, t)$ *is a thermodynamic process for* $\mathcal{B}$ *for all* $\boldsymbol{\lambda} \in \mathbb{E}^3$*, where*

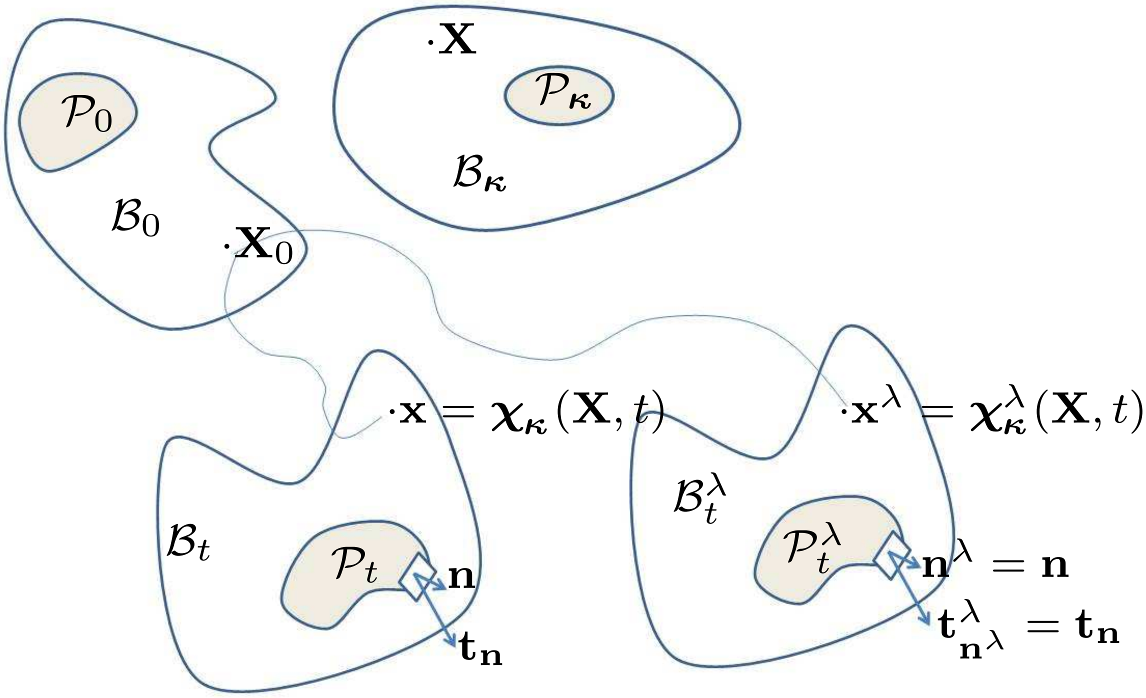
$$\pi^{\lambda}(\mathbf{X}, t) := \left\{ \boldsymbol{\chi}_{\boldsymbol{\kappa}}^{\lambda},\ \rho,\ \theta,\ \varepsilon,\ \eta,\ \mathbf{t_n},\ \mathbf{b},\ q_{\mathbf{n}},\ r \right\} (\mathbf{X}, t)\,,$$

with

$$\mathbf{x}^{\lambda} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}^{\lambda}(\mathbf{X}, t) := \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t) + \boldsymbol{\lambda} t = \mathbf{x} + \boldsymbol{\lambda} t\,.$$

Recall,

$$\mathcal{P}_t = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathcal{P}_{\boldsymbol{\kappa}}, t) \quad \text{and} \quad \mathcal{P}_t^{\lambda} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}^{\lambda}(\mathcal{P}_{\boldsymbol{\kappa}}, t) = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathcal{P}_{\boldsymbol{\kappa}}, t) + \boldsymbol{\lambda} t\,.$$



Axiom 1 requires

$$\dot{E}^\lambda(\mathcal{P}, t) + \dot{K}^\lambda(\mathcal{P}, t) = P^\lambda(\mathcal{P}, t) + Q^\lambda(\mathcal{P}, t)$$

and

$$\dot{H}^\lambda(\mathcal{P}, t) \geq S^\lambda(\mathcal{P}, t)$$

for all $\mathcal{P} \subset \mathcal{B}$ and for all $t \in \mathbb{R}^+$ whenever $\pi(\mathbf{X}, t)$ is a thermodynamic process for $\mathcal{B}$.

Global Integral Measures for $\pi^\lambda(\mathbf{X}, t)$

$$E^\lambda(\mathcal{P}, t) := \int_{\mathcal{P}_t^\lambda} \rho\varepsilon \, dv^\lambda = \int_{\mathcal{P}_t} \rho\, \varepsilon \, dv = E(\mathcal{P}, t) \, ,$$

$$\begin{aligned} K^\lambda(\mathcal{P}, t) &:= \int_{\mathcal{P}_t^\lambda} \frac{1}{2}\, \rho \left| \mathbf{v}^\lambda \right|^2 dv^\lambda \\ &= K(\mathcal{P}, t) + \boldsymbol{\lambda} \cdot \int_{\mathcal{P}_t} \rho\, \mathbf{v}\, dv + \frac{1}{2}\, |\boldsymbol{\lambda}|^2 \int_{\mathcal{P}_t} \rho \, dv \, , \end{aligned}$$

$$\begin{aligned} P^\lambda(\mathcal{P}, t) &:= \int_{\partial\mathcal{P}_t^\lambda} \mathbf{t_n} \cdot \mathbf{v}^\lambda \, da^\lambda + \int_{\partial\mathcal{P}_t^\lambda} \rho\mathbf{b} \cdot \mathbf{v}^\lambda \, dv^\lambda \\ &= P(\mathcal{P},\, t) + \boldsymbol{\lambda} \cdot \left(\int_{\partial\mathcal{P}_t} \mathbf{t_n} \, da + \int_{\mathcal{P}_t} \rho\mathbf{b} \, dv \right) \, , \end{aligned}$$

$$Q^\lambda(\mathcal{P},\, t) = Q(\mathcal{P},\, t)\, , \quad H^\lambda(\mathcal{P},\, t) = H(\mathcal{P},\, t)$$

and

$$S^\lambda(\mathcal{P},\, t) = S(\mathcal{P},\, t)\, .$$

Consequences of Axiom 1

$$\frac{1}{2}|\boldsymbol{\lambda}|^2 \underbrace{\frac{d}{dt}\int_{\mathcal{P}_t} \rho\, dv}_{=0}$$

$$+\boldsymbol{\lambda}\cdot\left(\underbrace{\frac{d}{dt}\int_{\mathcal{P}_t} \rho\mathbf{v}\, dv - \left(\int_{\partial\mathcal{P}_t} \mathbf{t_n}\, da + \int_{\mathcal{P}_t} \rho\mathbf{b}\, dv\right)}_{=0}\right) = 0$$

(i) **Local Balance of Mass** for $\pi(\mathbf{X}, t)$, $\forall\mathbf{X} \in \mathcal{B}_{\boldsymbol{\kappa}}$ and $t \in \mathbb{R}^+$:

$$\rho(\mathbf{X}, t) = \frac{\rho_0(\mathbf{X})|\det\nabla\boldsymbol{\kappa}_0(\mathbf{X})|}{|\det\nabla\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t)|}, \quad \rho_0(\mathbf{X}) \equiv \rho(\mathbf{X}, 0)$$

or, for $\tilde{\rho}(\mathbf{X}_0, t) := \rho(\mathbf{X}, t)|_{\mathbf{X}=\boldsymbol{\kappa}_0^{-1}(\mathbf{X}_0)}$ and $\forall\mathbf{X}_0 \in \mathcal{B}_0$ and $t \in \mathbb{R}^+$,

$$\tilde{\rho}(\mathbf{X}_0, t) = \frac{\tilde{\rho}(\mathbf{X}_0, 0)}{\det\nabla\boldsymbol{\chi}_0(\mathbf{X}_0, t)}$$

(ii) **Local Balance of Linear Momentum** for $\pi(\mathbf{X}, t)$: There exists a Cauchy stress tensor $\mathbf{T} : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^+ \mapsto$ Lin such that

$$\mathbf{t_n} = \mathbf{T}^{\mathsf{T}}\mathbf{n}$$

$$\text{div}\,\mathbf{T}^{\mathsf{T}} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}$$

(iii) **Local Balance of Energy for** $\pi(\mathbf{X}, t)$: The local balances of mass and linear momentum for $\pi(\mathbf{X}, t)$ yield the power theorem:

$$P(\mathcal{P},\, t) = \int_{\mathcal{P}_t} \mathbf{T}^{\mathsf{T}} \cdot \text{grad }\, \mathbf{v}\, dv + \dot{K}(\mathcal{P},\, t),$$

and with the balance of energy we may write, $\forall \mathcal{P} \subset \mathcal{B}$ and $\forall t \in \mathbb{R}^{+}$,

$$\int_{\partial \mathcal{P}_t} q_{\mathbf{n}}\, da = \int_{\mathcal{P}_t} \left[\rho \dot{\varepsilon} - \mathbf{T}^{\mathsf{T}} \cdot \text{grad}\, \mathbf{v} - \rho\, r\right]\, dv.$$

Consequently, there exists a heat flux vector $\mathbf{q} : \mathcal{B}_{\boldsymbol{\kappa}} \times \mathbb{R}^{+} \mapsto \mathbb{E}^{3}$ such that

$$\boxed{q_{\mathbf{n}} = -\mathbf{q} \cdot \mathbf{n}}$$

and

$$\boxed{\rho \dot{\varepsilon} = \mathbf{T}^{\mathsf{T}} \cdot \text{grad}\, \mathbf{v} - \text{div}\, \mathbf{q} + \rho\, r}$$

(iv) **Local Clausius-Duhem Inequality for** $\pi(\mathbf{X}, t)$: The Clausius-Duhem inequality and the local balances of mass and energy for $\pi(\mathbf{X}, t)$ yield

$$\boxed{\rho \dot{\eta} \geq -\text{div}\left(\frac{\mathbf{q}}{\theta}\right) + \rho \frac{r}{\theta}}$$

A Theorem: Thermodynamic Processes

Theorem 1 *Let $\mathcal{B}$ and $\mathcal{B}_{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\mathcal{B})$ be given with $\mathbf{X} \in \mathcal{B}_{\boldsymbol{\kappa}}$, and suppose $\pi(\mathbf{X}, t)$ is a thermodynamic process for $\mathcal{B}$. Define the collection*

$$\pi^*(\mathbf{X}, t) := \{\boldsymbol{\chi}^*_{\boldsymbol{\kappa}}, \rho^*, \theta^*, \varepsilon^*, \eta^*, \mathbf{t}^*_{\mathbf{n}^*}, \mathbf{b}^*, q^*_{\mathbf{n}^*}, r^*\}(\mathbf{X}, t)\,;$$

$$\mathbf{x}^* = \boldsymbol{\chi}^*_{\boldsymbol{\kappa}}(\mathbf{X}, t) := \mathbf{Q}(t)\,(\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, t) - \mathbf{o}) + \mathbf{c}(t)\,,$$

with $\mathbf{Q}(t) \in$ Orth, $\mathbf{c}(t) \in \mathbb{E}^3$ and $\mathbf{o} \in \mathbb{E}^3$ a fixed point, and

$$\{\rho^*, \theta^*, \varepsilon^*, \eta^*\}(\mathbf{X}, t) := \{\rho, \theta, \varepsilon, \eta\}(\mathbf{X}, t)\,,$$

$$\{\mathbf{t}^*_{\mathbf{n}^*},\, q^*_{\mathbf{n}^*}\}(\mathbf{X}, t) := \{\mathbf{Q}(t)\mathbf{t}_{\mathbf{n}},\, q_{\mathbf{n}}\}(\mathbf{X}, t)\,, \quad \text{with } \mathbf{n}^* := \mathbf{Q}(t)\mathbf{n}\,,$$

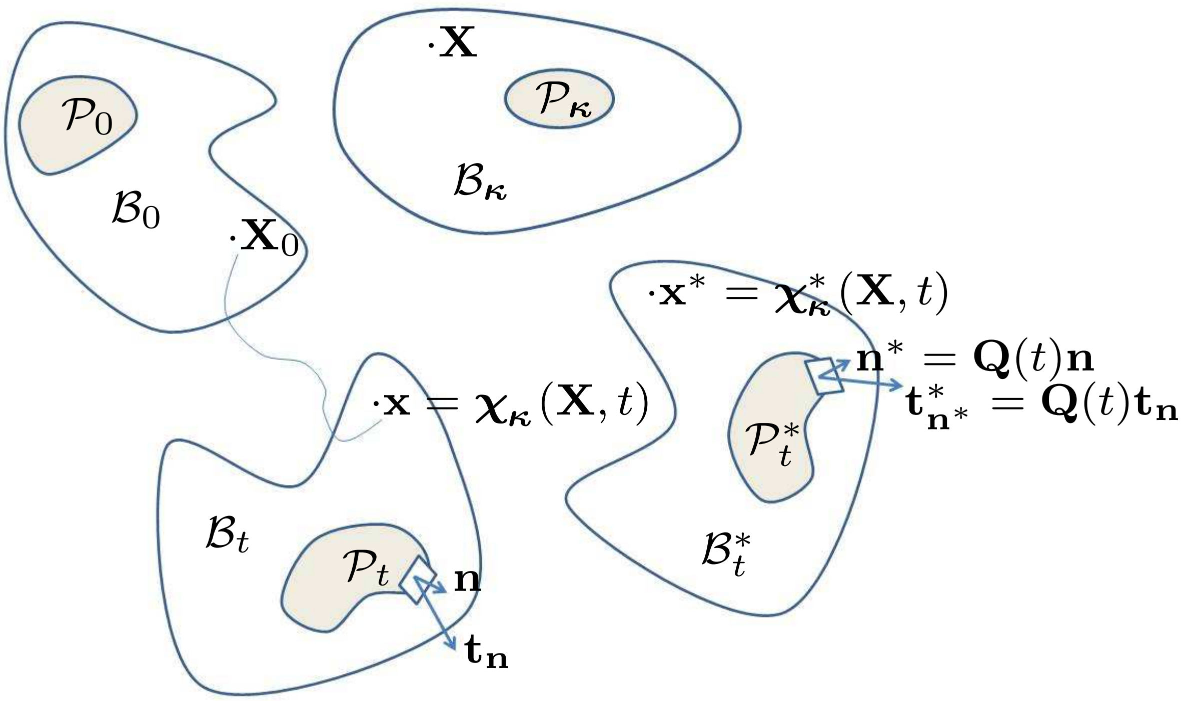
with $\{\mathbf{b}^*, r^*\}(\mathbf{X}, t)$ arbitrary. Then $\pi^*(\mathbf{X}, t)$ is a thermodynamic process for $\mathcal{B}$ if and only if:

$$\text{(1)}\quad \mathbf{b}^* = \mathbf{Q}\mathbf{b} + \ddot{\mathbf{c}} + \ddot{\mathbf{Q}}\,(\mathbf{x} - \mathbf{o}) + 2\dot{\mathbf{Q}}\mathbf{v}$$

$$\text{(2)}\quad r^* = r$$

$$\text{(3)}\quad \frac{d}{dt}\int_{\mathcal{P}_t} \rho\,(\mathbf{x} - \mathbf{o}) \times \mathbf{v}\, dv = \int_{\partial\mathcal{P}_t} (\mathbf{x} - \mathbf{o}) \times \mathbf{t}_{\mathbf{n}}\, da + \int_{\mathcal{P}_t} \rho\,(\mathbf{x} - \mathbf{o}) \times \mathbf{b}\, dv$$

for all $\mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$, where $\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X}, \mathbf{t})$ denotes a generic spatial point of $\mathcal{B}_t$.



$$\mathbf{v}^* := \dot{\boldsymbol{\chi}}^*_{\boldsymbol{\kappa}} = \dot{\mathbf{Q}}(\mathbf{x} - \mathbf{o}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}$$

$$\dot{\mathbf{v}}^* = \ddot{\boldsymbol{\chi}}^*_{\boldsymbol{\kappa}} = \ddot{\mathbf{Q}}(\mathbf{x} - \mathbf{o}) + 2\dot{\mathbf{Q}}\mathbf{v} + \mathbf{Q}\dot{\mathbf{v}} + \ddot{\mathbf{c}}$$

$$\text{grad}^*\mathbf{v}^* = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}(\text{grad}\,\mathbf{v})\mathbf{Q}^{\mathsf{T}}$$

To prove this theorem we need to show that $\pi^*(\mathbf{X}, t)$ satisfies

$$\dot{E}^*(\mathcal{P}, t) + \dot{K}^*(\mathcal{P}, t) = P^*(\mathcal{P}, t) + Q^*(\mathcal{P}, t)$$

$$\dot{H}^*(\mathcal{P}, t) \geq S^*(\mathcal{P}, t)$$

for all $\mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$, if and only if the conditional items (1), (2) and (3) hold.

Identities: Given a Thermodynamic Process $\pi(\mathbf{X}, t)$

Local balances of mass and linear momentum yields

$$(\mathrm{I})_1 : \quad \frac{d}{dt}\int_{\mathcal{P}_t} \rho\,(\mathbf{x}-\mathbf{o})\otimes\mathbf{v}\,dv - \left(\int_{\partial\mathcal{P}_t} (\mathbf{x}-\mathbf{o})\otimes\mathbf{t_n}\,da + \int_{\mathcal{P}_t} \rho\,(\mathbf{x}-\mathbf{o})\otimes\mathbf{b}\,dv\right) = \int_{\mathcal{P}_t} (\rho\mathbf{v}\otimes\mathbf{v}-\mathbf{T})\;dv$$

Local balances of mass, linear momentum and energy yields

$$(\mathrm{I})_2 : \quad \dot{E}^*(\mathcal{P},t) + \dot{K}^*(\mathcal{P},t) - (P^*(\mathcal{P},t) + Q^*(\mathcal{P},t)) = -\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q}\cdot\int_{\mathcal{P}_t}\mathbf{T}\,dv - \int_{\mathcal{P}_t}\rho\,(r^*-r)\;dv - \int_{\mathcal{P}_t}\rho\left(\mathbf{b}^* - \left(\mathbf{Q}\mathbf{b}+\ddot{\mathbf{c}}+\ddot{\mathbf{Q}}(\mathbf{x}-\mathbf{o})+2\dot{\mathbf{Q}}\mathbf{v}\right)\right)\cdot\mathbf{v}^*dv$$

Clausius-Duhem inequality for $\pi(\mathbf{X}, t)$ yields

$$(\mathrm{I})_3 : \quad \dot{H}^*(\mathcal{P},t) - S^*(\mathcal{P},t) = \underbrace{\dot{H}(\mathcal{P},t) - S(\mathcal{P},t)}_{\geq 0} - \int_{\mathcal{P}_t}\rho\left(\frac{r^*-r}{\theta}\right)\,dv$$

Proof:Sufficiency of (1), (2) and (3)

Suppose (1), (2) and (3) of Theorem 1 holds, i.e.,

(1) $\mathbf{b}^* = \mathbf{Q}\mathbf{b} + \ddot{\mathbf{c}} + \ddot{\mathbf{Q}}(\mathbf{x} - \mathbf{o}) + 2\dot{\mathbf{Q}}\mathbf{v}$

(2) $r^* = r$

(3) $$\frac{d}{dt}\int_{\mathcal{P}_t} \rho(\mathbf{x} - \mathbf{o}) \times \mathbf{v}\, dv = \int_{\partial\mathcal{P}_t} (\mathbf{x} - \mathbf{o}) \times \mathbf{t_n}\, da + \int_{\mathcal{P}_t} \rho(\mathbf{x} - \mathbf{o}) \times \mathbf{b}\, dv$$

Then, using (I)$_1$,

$$(3) \Longleftrightarrow \int_{\mathcal{P}_t} \mathbf{T}\, dv \in \text{Sym}$$

and, using (I)$_2$ and (I)$_3$ we see that (1), (2) and (3) $\Longrightarrow$

$$\dot{E}^*(\mathcal{P}, t) + \dot{K}^*(\mathcal{P}, t) = P^*(\mathcal{P}, t) + Q^*(\mathcal{P}, t)$$
$$\dot{H}^*(\mathcal{P}, t) \geq S^*(\mathcal{P}, t)$$

for all $\mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$.

Proof: Necessity of (1)

Suppose $\pi^*(\mathbf{X},t)$ is a thermodynamic process for $\mathcal{B}$. Then, Axiom 1 $\Longrightarrow \pi^*(\mathbf{X},t)$ satisfies the balance of linear momentum $\forall \mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$, i.e.,

$$\frac{d}{dt}\int_{\mathcal{P}_t^*} \rho^* \mathbf{v}^* \, dv^* = \int_{\partial \mathcal{P}_t^*} \mathbf{t}^*_{\mathbf{n}^*} \, da^* + \int_{\mathcal{P}_t^*} \rho^* \mathbf{b}^* \, dv^*$$

Now, use transformation hypotheses for $\pi^*(\mathbf{X},t)$ and balance of linear momentum for $\pi(\mathbf{X},t)$ to get

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}_t} \rho \left(\overbrace{\dot{\mathbf{Q}}(\mathbf{x}-\mathbf{o}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}}^{\mathbf{v}^*} \right) dv \\ = \mathbf{Q}\int_{\partial \mathcal{P}_t} \mathbf{t}_{\mathbf{n}} \, da + \int_{\mathcal{P}_t} \rho \mathbf{b}^* \, dv \\ = \mathbf{Q}\frac{d}{dt}\int_{\mathcal{P}_t} \rho \mathbf{v} \, dv + \int_{\mathcal{P}_t} \rho \left(\mathbf{b}^* - \mathbf{Q}\mathbf{b} \right) dv\end{aligned}$$

Then, use balance of mass for $\pi(\mathbf{X},t)$ to reach

$$\int_{\mathcal{P}_t} \rho \left(\mathbf{b}^* - \left(\mathbf{Q}\mathbf{b} + \ddot{\mathbf{c}} + \ddot{\mathbf{Q}}(\mathbf{x}-\mathbf{o}) + 2\dot{\mathbf{Q}}\mathbf{v} \right) \right) dv = \mathbf{0}$$

$\forall \mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$. This proves (1).

Proof: Necessity of (2) and (3)

Suppose $\pi^*(\mathbf{X},t)$ is a thermodynamic process. Then, (1) and $(\mathrm{I})_2$ yield

$$\begin{aligned}&\dot{E}^*(\mathcal{P},t)+\dot{K}^*(\mathcal{P},t)-(P^*(\mathcal{P},t)+Q^*(\mathcal{P},t))\\ &= \quad -\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q}\cdot\int_{\mathcal{P}_t}\mathbf{T}\,dv-\int_{\mathcal{P}_t}\rho\,(r^*-r)\,dv\end{aligned}$$

Then, because $\pi^*(\mathbf{X},t)$ satisfies the balance of energy, we have

$$\int_{\mathcal{P}_t}\left(\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q}\cdot\mathbf{T}+\rho\,(r^*-r)\right)\,dv=0\quad\forall\mathcal{P}\subset\mathcal{B}\quad\forall t\in\mathbb{R}^+$$

which implies

$$\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q}\cdot\mathbf{T}+\rho\,(r^*-r)=0$$

Now, because $\pi^*(\mathbf{X},t)$ satisfies the Clausius-Duhem inequality, we see that $(\mathrm{I})_3$ requires

$$\underbrace{\dot{H}(\mathcal{P},t)-S(\mathcal{P},t)}_{\geq 0}-\int_{\mathcal{P}_t}\rho\left(\frac{r^*-r}{\theta}\right)\,dv\geq 0$$

which may be replaced by

$$\underbrace{\dot{H}(\mathcal{P},t)-S(\mathcal{P},t)}_{\geq 0}+\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q}\cdot\int_{\mathcal{P}_t}\frac{\mathbf{T}}{\theta}\,dv\geq 0$$

Proof: Necessity of (2) and (3)... continued

Thus, we have

$$\underbrace{\dot{H}(\mathcal{P},t) - S(\mathcal{P},t)}_{\geq 0} + \dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q} \cdot \int_{\mathcal{P}_t} \frac{\mathbf{T}}{\theta}\, dv \geq 0$$

which is supposed to hold for the thermodynamic process $\pi(\mathbf{X}, t)$, for all $\mathbf{Q}(t) \in$ Orth and for all $\mathcal{P} \subset \mathcal{B}$ and $t \in \mathbb{R}^+$.

Now, fix $t = t_0$ and choose $\mathbf{Q}(t)$ such that $\mathbf{Q}(t_0) = \mathbf{1}$ and $\dot{\mathbf{Q}}^{\mathsf{T}}(t_0) = \mathbf{W}$, where $\mathbf{W} \in$ Skew is arbitrarily specified. Then the inequality can be violated at $t = t_0$ unless

$$\mathbf{W} \cdot \int_{\mathcal{P}_{t_0}} \frac{\mathbf{T}}{\theta}\, dv = 0 \quad \forall \mathbf{W} \in \text{Skew}$$

Since $\mathcal{P} \subset \mathcal{B}$ and t_0 are arbitrary we conclude that

$$\mathbf{T} \in \text{Sym}$$

Thus,

- (I)$_1$ $\Longrightarrow$ (3) holds.
- $\dot{\mathbf{Q}}^{\mathsf{T}}\mathbf{Q} \cdot \mathbf{T} + \rho\,(r^* - r) = 0 \Longrightarrow r = r^*$ so that (2) holds.

Euclidean Invariance

Axiom 2 (Euclidean Invariance) *For each* $\pi(\mathbf{X},t)$ *that is a thermodynamic process for* $\mathcal{B}$*, there exists a pair* $\{\mathbf{b}^*, r^*\}(\mathbf{X},t)$ *such that the collection*

$$\pi^*(\mathbf{X},t) := \{\boldsymbol{\chi}^*_{\boldsymbol{\kappa}}, \rho^*, \theta^*, \varepsilon^*, \eta^*, \mathbf{t}^*_{\mathbf{n}^*}, \mathbf{b}^*, q^*_{\mathbf{n}^*}, r^*\}(\mathbf{X},t)$$

is a thermodynamic process for $\mathcal{B}$ *for all* $\mathbf{Q}(t) \in$ Orth *and* $\mathbf{c}(t) \in \mathbb{E}^3$*, where*

$$\boldsymbol{\chi}^*_{\boldsymbol{\kappa}}(\mathbf{X},t) := \mathbf{Q}(t)\left(\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\mathbf{X},t) - \mathbf{o}\right) + \mathbf{c}(t)$$

with $\mathbf{o} \in \mathbb{E}^3$ *a fixed point,*

$$\{\rho^*, \theta^*, \varepsilon^*, \eta^*\}(\mathbf{X},t) := \{\rho, \theta, \varepsilon, \eta\}(\mathbf{X},t)$$

and

$$\{\mathbf{t}^*_{\mathbf{n}^*},\, q^*_{\mathbf{n}^*}\}(\mathbf{X},t) := \{\mathbf{Q}(t)\mathbf{t}_{\mathbf{n}},\, q_{\mathbf{n}}\}(\mathbf{X},t)\,, \quad \text{with } \mathbf{n}^* := \mathbf{Q}(t)\mathbf{n}\,.$$

Remarks

(1) Axiom 2 $\not\Rightarrow$ Axiom 1: $\{\mathbf{b}^*, r^*\}(\mathbf{X}, t)$ is not specified in Axiom 2.

(2) Axiom 1 + Axiom 2 $\Longrightarrow$

(i) $\mathbf{b}^* = \mathbf{Q}\mathbf{b} + \ddot{\mathbf{c}} + \ddot{\mathbf{Q}}\,(\mathbf{x} - \mathbf{o}) + 2\dot{\mathbf{Q}}\mathbf{v}$

(ii) $r^* = r$

(iii) The balance of moment of momentum holds for any thermodynamic process $\pi(\mathbf{X}, t)$ for $\mathcal{B}$.

Objectivity

Axiom 2 and the consequences of Axiom 1 shows that there exists a Cauchy stress tensor $\mathbf{T}^* = \mathbf{T}^*(\mathbf{X}, t)$ and a heat flux vector $\mathbf{q}^* = \mathbf{q}^*(\mathbf{X}, t)$ for $\pi^*(\mathbf{X}, t)$ such that

$$\mathbf{t}^*_{\mathbf{n}^*} = \mathbf{T}^{*\mathsf{T}}\mathbf{n}^* \,, \quad q^*_{\mathbf{n}^*} = -\mathbf{q}^* \cdot \mathbf{n}^*$$

for all unit vectors $\mathbf{n}^*$. Axiom 2 then requires

$$\mathbf{T}^*(\mathbf{X}, t) = \mathbf{Q}(t)\mathbf{T}(\mathbf{X}, t)\mathbf{Q}^{\mathsf{T}}(t) \,, \quad \mathbf{q}^*(\mathbf{X}, t) = \mathbf{Q}(t)\mathbf{q}(\mathbf{X}, t)$$

for all $\mathbf{X} \in \mathcal{B}_{\boldsymbol{\kappa}}$, all $t \in \mathbb{R}^+$ and all $\mathbf{Q}(t) \in$ Orth. Thus, the Cauchy stress tensor and the heat flux vector are said to be “objective”.